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Monte Carlo investigation of the three-dimensional random-field three-state Potts model

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Abstract. The three-state Potts model with nearest ferromagnetic interaction J on the simple cubic lattice, exposed to a field $H_{RF} = J$ which randomly favours one of the states, is studied by finite size scaling analysis of Monte Carlo simulations. The consequences of the hyperscaling violations for the finite size scaling description are worked out, and it is shown that at criticality the order parameter distribution develops a trivial character (superposition of three delta functions), unlike the nontrivial distributions in models where hyperscaling holds. The numerical data are compatible with this interpretation, and are also compatible with a description in terms of only two independent static critical exponents, as proposed by Schwartz *et al*. Slow relaxation and the need of averaging over a large number of random field samples restricts the range of linear dimensions L to $L \leqslant 28$, and in view of crossover problems only rough estimates of critical exponents are obtained.

1. Introduction

The effect of quenched random fields on phase transitions has been a long-standing challenge in statistical mechanics (Imry and Ma 1975, Nattermann and Villain 1988, Nattermann and Rujan 1989, Belanger and Young 1991, Rieger 1995). The generic model is the nearest neighbour random field Ising model (RFIM), for which—after a long controversy!—it was proven that in $d = 3$ dimensions at low temperatures a spontaneous magnetization exists (Imbrie 1984, Bricmont and Kupiainen 1987). However, the nature of the transition from the ferromagnetic to the disordered phase is still incompletely understood: *assuming* that this is a second-order transition there are important questions about the nature of the critical behaviour (Schwartz *et al* 1991, Gofman *et al* 1993, Rieger and Young 1993, Rieger 1995). But a weak first-order transition (Young and Nauenberg 1985) is not really ruled out (Rieger 1995). Recently there has been new evidence for the old idea (Morgenstern *et al* 1981) that on cooling the system from the disordered paramagnetic phase one first enters a glass-like phase (de Almeida and Bruinsma 1987, Mézard and Young 1992, Mézard and Monasson 1994, de Dominicis *et al* 1995, Stepanow *et al* 1996).

The situation is even more unclear when one considers the q-state Potts model (Potts 1952, Wu 1982) exposed to random fields (Blankschtein *et al* 1984, Goldschmidt and Xu 1985, 1986, Eichhorn and Binder 1995, 1996). This model is of interest because of a possible application to anisotropic orientational glasses (Binder and Reger 1992, Vollmayr *et al* 1992), e.g. in a coarse-grained description of systems such as $(N_2)_x Ar_{1-x}$ (Hoechli *et al* 1990) the q discrete states of the Potts model may represent the q energetically preferred orientations of the quadrupole moment of the N_2 molecule in the crystal field potential, and the random field represents the influence of the disorder in the environment caused

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by random dilution (with Ar in this example). But the main interest, of course, comes from the fact that for a Potts ferromagnet the *order* of its transition to the disordered phase depends on the number of its states q (Wu 1982). If we consider q, d as continuous rather than discrete variables, we may draw a line $q_c(d)$ in the q–d plane, where the transition changes from first-order to second-order, figure 1. In the presence of random fields, this line of tricritical points (Sarbach and Lawrie 1984) gets shifted, since both the lower critical dimensionality d_ℓ (Fisher 1974) and the upper critical dimensionality d_u get shifted due to the random fields Aharony *et al* 1976). Remember that d_ℓ is the dimensionality where thermal fluctuations destroy long range order (Young 1980), while for $d > d_u$ meanfield theory becomes qualitatively correct, and $d_u = 4$ for the pure system while $d_u = 6$ in the presence of random fields (Aharony *et al* 1976, Blankschtein *et al* 1984). Based on a perturbational renormalization group treatment, Aharony *et al* (1976) suggested the concept of simple 'dimensional reduction', i.e. the random field system corresponds to a pure system in a dimensionality shifted by 2, $d_{RF} = d_{FM} + 2$. This would imply the broken curve in figure 1, which cannot be correct since it implies $d_\ell = 3$ (note $q_c(d \to d_\ell) \to \infty$), while it is known both from droplet model type arguments (Villain 1982, Grinstein and Ma 1982) and rigorous analysis (Imbrie 1984) that $d_\ell = 2$ in the presence of random fields. But conceivably a modified dimensional reduction holds (Schwartz 1985, 1991, Schwartz and Soffer 1985, 1986), such that a system at dimensionality d_{RF} in the presence of random fields has the same critical behaviour as a pure ferromagnet at dimensionality d_{FM}, with

$$d_{RF} = d_{FM} + 2 - \eta(d_{FM}) \tag{1}$$

where the exponent η characterizes the decay of critical correlations (Fisher 1974). Since $\eta(d = 1) = 1$ for pure Ising models, equation (1) yields $d_{RF} = 2$ for $d_{FM} = 1$ as the lower critical dimensionality. However, even assuming that there is a second-order transition from the ferromagnetic phase to the disordered phase in the presence of the random field: Villain (1985) and Fisher (1986) suggest that there are three independent static exponents; e.g. the hyperscaling relation (Fisher 1974) $\gamma(d_{FM}) + 2\beta(d_{FM}) = d_{FM}\nu(d_{FM})$ gets modified as (β, γ, ν are the critical exponents of order parameter, susceptibility and correlation length, see e.g. Fisher (1974))

$$\gamma + 2\beta = \nu(d - \theta). \tag{2}$$

Here θ is a new, independent exponent (while $2 - \eta = \gamma/\nu$ remains valid). According to Blankschtein *et al* (1984), Schwartz (1985) and Schwartz *et al* (1991), on the other hand, standard hyperscaling does not hold either, but the exponent θ in equation (2) is not an independent exponent:

$$\theta = 2 - \eta. \tag{3}$$

Blankschtein *et al* (1984) have proposed that a modified dimensional reduction applies to the q-state Potts model, implying that the separatrix between first- and second-order transitions (that starts at $q = 2$, $d = 4$ for the pure system and tends to $q \to \infty$ for $d \to 1$) gets transformed into a curve that starts at $q = 2$, $d = 6$ for the random field system and tends to $q \to \infty$ for $d \to 2$. The physically most relevant case $q = 3$, $d = 3$, which is a weak first-order transition in the pure case (Herrmann 1979, Stepanow and Tsypin 1991, Kikuchi and Okabe 1992, Vollmayr *et al* 1993), falls in the second-order regime according to figure 1 if random fields are present.

If one accepts that figure 1 is valid, the next questions arise as follows: will arbitrary weak random fields suffice to turn the transition from first- to second-order (this is believed to occur in $d = 2$ for $q > 4$ (Aizenman and Weber 1989)), or do we need a finite strength of the random field H_{RF} (figure 2(*b*), Eichhorn and Binder 1995), while an alternative, third

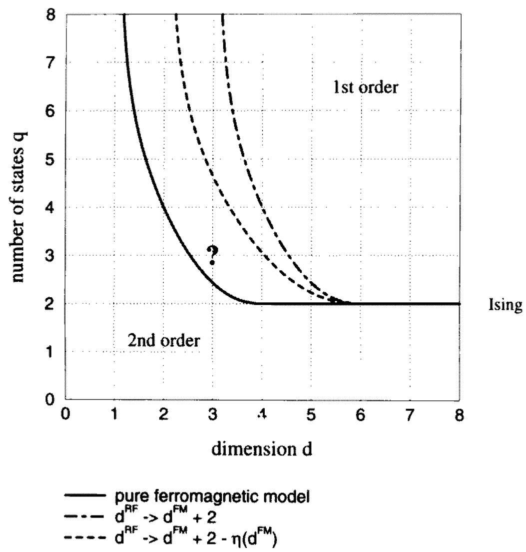


Figure 1. Schematic plot of the separatrix $q_c(d)$ in the q–d plane. Potts models below this curve have second-order transitions, above the curve they have first-order transition. The full curve refers to the pure case, while the dash–dotted curve refers to the random field Potts model (RFPM). The broken curve shows the result that one would obtain when the concept of simple dimensional reduction would apply (i.e. the 'effective dimensionality' d^{RF} of the random field model corresponds to a pure ferromagnet raised by two, $d^{FM}+2$). Remember that $q = 2$ simply corresponds to the Ising model. The question mark in figure 1 emphasizes the physically most relevant case $d = 3$, $q = 3$. After Blankschtein *et al* (1984).

scenario is given in figure 2(a) (Eichhorn and Binder 1996). The latter scenario should be applied if the claims of Goldschmidt and Xu (1985, 1986) are correct, who suggested that the phase transition of the three-state Potts model stays first-order throughout the whole (H_{RF},T) plane. Figure 2(b) is compatible with the dimensionality shift scenario while figure 2(a) clearly is not. However, while figure 2 is compatible with a simple mean field treatment of the RFIM (Aharony 1978), both versions disregard the possibility of first-order transitions in weak random fields (Young and Nauenberg 1985) and glassy phases (Mézard and Monasson 1994, de Dominicis *et al* 1995).

In the present paper we try to contribute to the understanding of this problem by a Monte Carlo study of the three-state Potts model in weak random fields. Previous work (Eichhorn and Binder 1996) applying fields of intermediate strength gave evidence for rather dramatic slowing down near the transition, and its character could not be clarified (if it is first-order, it is rather weakly first-order only). Here we use finite size scaling methods (Fisher 1971, Binder 1981, Barber 1983, Privman 1991, Binder 1992), but pay particular attention to the hyperscaling violation (equation (2)) and its consequences: as is well known, finite size scaling in its usual form implies the validity of hyperscaling (Binder 1981, Brézin 1982, Zinn-Justin and Brézin 1985). However, using both equations (2) and (3) finite size scaling is recovered, if the probability distribution of the order parameter has a particular (rather pathological!) form (i.e., a sum of delta functions in the scaling limit). It is shown that the actual Monte Carlo data are reasonably compatible with such an analysis (a brief and preliminary account of these results was given in Eichhorn and Binder 1995). The implication of our analysis is that the Potts ferromagnet in random fields (which are neither

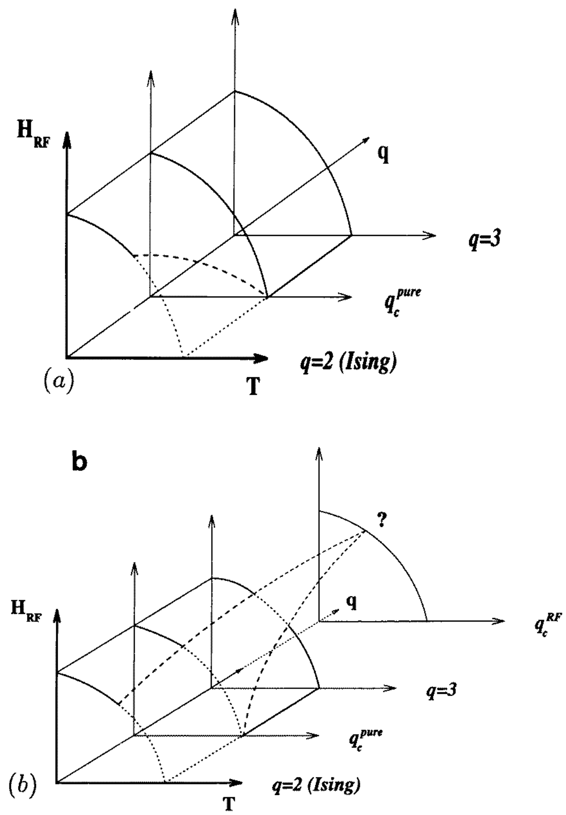


Figure 2. Schematic phase diagram scenarios for the q-state Potts model in $d = 3$ dimensions, using the temperature T, the amplitude of the random field H_{RF} and the number of Potts-states q as variables. Second-order transitions are shown as dotted lines, while first-order transitions are shown as full lines, tricritical points being shown as broken lines. In the Ising model ($q = 2$), the transition is second-order for small H_{RF} and first-order for large H_{RF}. In case (a) it is assumed that the line of tricritical points ends for $q = q_c^{pure}$, implying that only first-order transitions occur for $q > q_c^{pure}$ (consistent with Goldschmidt and Xu 1985, 1986). Conversely, in case (b) it is assumed that for $q = q_c^{pure}$ a line of other tricritical points start at $H_{RF} = 0$, $q = q_c^{pure}$ such that for $q_c^{pure} < q < q_c^{RF}$ two tricritical points occur along the phase transition line $H_c = H_{RF}(T_c)$ from the ordered to the disordered state. Note that q_c^{RF} can be considered as the value of q where these two tricritical points merge and annihilate each other.

too weak nor too strong!) has a second-order transition and the exponents are presumably compatible with the modified dimensionality reduction proposed by Schwartz (1985) and Schwartz *et al* (1991), although due to crossover problems we are unable to give precise estimates of the associated exponents.

In section 2 we briefly discuss our Monte Carlo methods (see also Eichhorn 1995, Eichhorn and Binder 1996), while section 3 explores the consequences of hyperscaling violation on finite size scaling. Section 4 then summarizes our numerical results, while section 5 briefly summarizes our conclusions.

2. Some comments on the simulation technique

Due to the need of sample averaging and very slow relaxation the simulation of systems with quenched disorder is still a challenge (Young *et al* 1995, Rieger 1995). Applying multispin coding techniques (see e.g. Bhanot *et al* 1986) a reasonably fast program was developed (Eichhorn 1995), operating at a speed of about 2×10^7 Monte Carlo steps (MCS) per second on a CRAY-YMP processor, and 1.3×10^6 MCS sec^{-1} on an INTEL PARAGON I860 processor. These numbers refer to a conventional single spin flip Metropolis type algorithm (Binder and Heermann 1992). Since the 'quenched averaging' (Binder and Young 1986) over the random field configurations can be done trivially in parallel, we have found it efficient to run our program on an INTEL PARAGON multiprocessor machine simply using each available processor for a replica of the system with a different random field configuration. For small systems (from $L = 4$ to $L = 12$) we typically carried out an averaging over 496 realizations of the random field, while for the largest size presented here ($L = 28$) up to 248 realization were used.

Typically we equilibrated our systems for times of order of 10^5 to 10^6 MCS per lattice site. As is well known, for random systems it is a severe problem to judge whether a system has reached full thermal equilibrium or is still in a slowly relaxing metastable state (Binder and Young 1986, Bhatt and Young 1988). For our problem we have found it useful to study the relaxation of magnetization m and energy E as a function of time, using different starting conditions (one replica (A) was started in a fully ordered state, the other replica (B) was started from a completely random state). Figure 3 shows that in the critical region a remarkably fast relaxation towards equilibrium is obtained for $H_{RF} = 1$, while for $H_{RF} = 2$ the relaxation is several orders of magnitude slower, which is evident from an investigation of the relaxation time in the critical region (figure 4). As a consequence, we have confined ourselves to a careful study of the case $H_{RF} = 1$ and no further work is done for $H_{RF} = 2$ (note that we use only integer values of H_{RF}, in units of the exchange constant, since then our program performs somewhat faster than for noninteger choices, see Eichhorn (1995)).

We tried to analyse the relaxation times in figure 4 in terms of appropriate concepts about critical slowing down. While for ordinary critical phenomena in the absence of conservation laws one would postulate a simple power law with a dynamic exponent z, see Hohenberg and Halperin (1977)

$$\tau = a_1(T/T_c - 1)^{-z} \qquad T \to T_c \quad \text{from above} \tag{4}$$

in the case of RFIM also a stronger divergence of the relaxation time is under discussion, namely 'activated dynamics' (Villain 1985, Fisher 1986)

$$\ln\tau = a_2(T/T_c - 1)^{-z'} \qquad T \to T_c \quad \text{from above.} \tag{5}$$

Of course, in a finite $L \times L \times L$ box either divergence would exhibit some finite size rounding, and hence small lattice sizes systematically yield smaller estimates for τ, as is obvious from the figure. Curves in figure 4 represent fits to the largest size only. Obviously, there is still much scatter in the data, and hence we cannot attempt to distinguish between the two conflicting possibilities, equations (4) and (5). Moreover, it is obvious that τ is already extremely large for $H_{RF} = 2$ even for rather small values of L. That is why we decided to analyse here only the case $H_{RF} = 1$ in some detail. Obviously, a study of the critical dynamics of the RFIM and RFPM would be extremely interesting, but this would require many more orders of magnitude of CPU time than were available here (we have used the equivalent of several thousands hours of CPU time on a CRAY-YMP processor already).

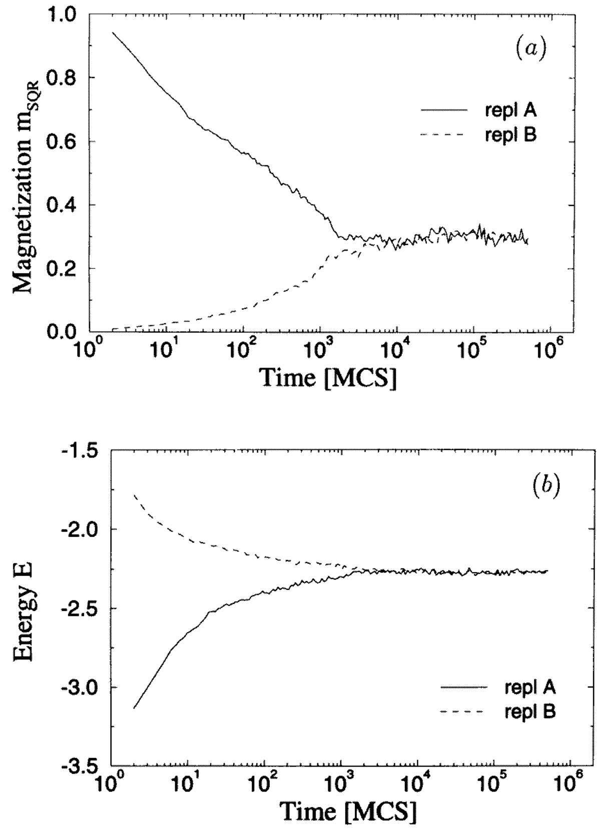


Figure 3. Relaxation of the magnetization (a) and energy (b) versus time t (in units MCS), for $L = 28$, $T = 1.695$, $H_{RF} = 1$, averaging over 62 realizations of the random field. Note $k_B = 1$, and the nearest neighbour exchange $J = 1$. Both $k_B T$ and H_{RF} are measured in units of J.

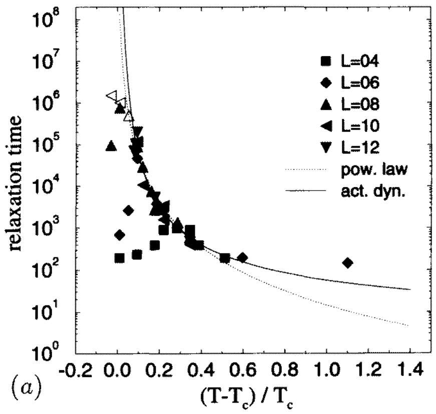
We have typically carried out runs for a small number of temperatures only (between 3 and 9 values) and record the quantities of interest every $10^4 (L \leqslant 6)$ or $2 \times 10^4 (L \geqslant 8)$ MCS per spin, taking between 25 000 and 50 000 such 'measurements' for each of our 248 random field samples. In this way histograms for the low-order moments $\overline{m^p}(E_i)$ are taken individually for each random field sample. Here E_k is the energy recorded in the kth observation, and noting that in our model only discrete energies E_i can occur, h_i is the energy distribution ('histogram', cf. Ferrenberg and Swendsen 1988, 1989),

$$h_i = \sum_k \delta(E_k - E_i) \qquad \overline{m_i^p} = \sum_k m_k^p \delta(E_k - E_i) \qquad p = 1, 2, 3, 4. \tag{6}$$

If we denote the temperature where the simulation is carried out by T_o, the thermal expectation value at this temperature is

$$\langle m^p \rangle_{T_o} = \sum_i h_i \overline{m_i^p} \Big/ \sum_i h_i \tag{7}$$

and the expectation value at a neighbouring temperature T in this 'single histogram



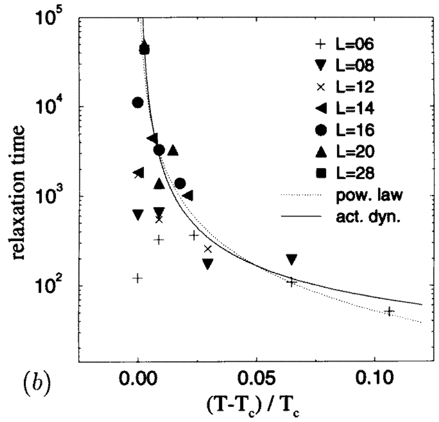


Figure 4. Relaxation time τ (estimated as the time where the magnetization m became independent of the initial condition, as shown in figure 3(a)) plotted against $T - T_c$, for $H_{RF} = 2$ (a) and $H_{RF} = 1$ (b), using the estimates $T_c = 1.19$ (a) and $T_c = 1.69$ (b). Dotted curves are fits to a power law, $\tau = a_1(T/T_c - 1)^{-z}$, and full curve to a thermally activated critical slowing down, $\ln \tau = a_2(T/T_c - 1)^{-z'}$. Parameters of the fit are $a_1 = 14.4$, $z = 3.49$ or $a_2 = 4$, $z' = 0.43$ in case (a), and $a_1 = 1.08$, $z = 1.68$ or $a_2 = 2.48$, $z' = 0.25$ in case (b). Note that the open symbols in case (a) have the meaning of lower bounds only, since the runs were simply too short to reach thermal equilibrium for these parameter combinations. Different linear dimensions are included, as indicated in the figure.

extrapolation' (Ferrenberg and Swendsen 1988) becomes

$$\langle m^p \rangle_T = \sum_i h_i \exp\left[\left(\frac{1}{T_o} - \frac{1}{T}\right) E_i\right] \overline{m_i^p} \Big/ \sum_i h_i \exp\left[\left(\frac{1}{T_o} - \frac{1}{T}\right) E_i\right]. \tag{8}$$

This reweighting technique is readily generalized to take into account the information of multiple histograms taken at a set of several temperatures (Ferrenberg and Swendsen 1989). For systems with quenched random disorder, of course, there is the difficulty that the thermal averaging (such as in equiation (8)) has to be carried out for each sample of the random disorder individually (D'Onorio De Meo *et al* 1995), and only afterwards can one take the average $[\ldots]_{av}$ over the quenched random disorder (Binder and Young 1986). Thus the storage requirements for reweighting techniques applied to random systems are huge—even restricting attention to $p \leqslant 4$ in equation (6) we have needed 1.25 MByte working storage per random field realization. Figure 5 shows typical data obtained with such techniques (symbols are only drawn at temperatures where runs have actually been carried out for the respective lattice sizes). We shall return to the analysis of these data in Sec. 4. Here the energy E is simply the thermal average of the RFPM Hamiltonian (remember $J = 1$, $H_{RF} = 1$)

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta_{S_i, S_j} - H_{RF} \sum_i \delta_{S_i h_i} \qquad S_i = 1, 2, \ldots, q \tag{9}$$

where the quenched random variable h_i is chosen uniformly and randomly from the set $\{1, 2, \ldots, q\}$. The specific heat per spin then is sampled from the fluctuation relation

$$C_v = ([\langle \mathcal{H}^2 \rangle_T - \langle \mathcal{H} \rangle_T^2]_{av})/(T^2 L^d). \tag{10}$$

For the order parameter $\boldsymbol{m}$, it is convenient to use the simplex representation of the Potts model (Vollmayr *et al* 1993). For $q = 3$, this means we have a two-component order parameter $\boldsymbol{m} = (m_1, m_2)$, where the individual spins can be one of the following three unit vectors

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{e}_2 = \begin{pmatrix} -1/2 \\ +\sqrt{3}/2 \end{pmatrix} \qquad \boldsymbol{e}_3 = \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix}. \tag{11}$$

If we then denote the Potts spin vector at lattice site i as $\boldsymbol{s}_i$ ($\boldsymbol{s}_i$ can be one of these three discrete vectors in equation (11) only, of course), we can define a $(q-1)$ dimensional magnetization vector $\boldsymbol{m}$ as

$$\boldsymbol{m} = (1/N)\sum_i \boldsymbol{S}_i \qquad m_{SQR} = [\langle|\boldsymbol{m}|\rangle_T]_{av}. \tag{12}$$

The 'susceptibility' χ' shown in figure 5(d) is then defined in terms of a fluctuation relation

$$\chi' = L^d[\langle \boldsymbol{m}^2\rangle_T - \langle|\boldsymbol{m}|\rangle_T^2]_{av}/T. \tag{13}$$

Note that we use $\langle|\boldsymbol{m}|\rangle_T$ rather than $\langle\boldsymbol{m}\rangle_T$ in equation (13) because $\langle\boldsymbol{m}\rangle_T \equiv 0$ in finite systems in the absence of symmetry breaking fields (Binder and Heermann 1992). Therefore χ will differ above T_c from the susceptibility $\partial[\langle m_1\rangle/\partial H_1]_T$, where H_1 is a uniform field applied in the direction of m_1, by a constant factor. But χ' as defined in equation (13) has the advantage that also in the absence of random and/or uniform fields there is a well defined peak in a finite system present (Binder and Heermann 1992).

Due to the presence of the random field there is also interest in a 'disconnected susceptibility' χ_{dis} (Schwartz 1985, Schwartz and Soffer 1985, 1986)

$$T\chi_{dis} = L^d[\langle\boldsymbol{m}\rangle_T^2]_{av} \tag{14}$$

which differs in its critical behaviour from χ' or χ which usually is defined as follows

$$T\chi = L^d[\langle\boldsymbol{m}^2\rangle_T - \langle\boldsymbol{m}\rangle_T^2]_{av}. \tag{15}$$

3. Finite size scaling for random field systems

In order to appreciate why equation (2) creates a problem for finite size scaling, we recall a phenomenological connection between finite size scaling and hyperscaling (i.e. $\mathrm{d}\nu = 2\beta + \gamma$, see Fisher (1974), as derived by Binder (1981)). There one considers the distribution function $P_L(\boldsymbol{m})$ for the order parameter in a finite system of linear dimensions L in the critical region, where the correlation length ξ of order parameter fluctuations is large. In the absence of fields, approach to criticality by variation of the temperature distance $1 - T/T_c$ can then be described by variation of $\xi \propto |1 - T/T_c|^{-\nu}$. Finite size scaling then simply is the statement that $P_L(\boldsymbol{m})$ does not depend on the three variables L, ξ, $\boldsymbol{m}$ separately, but is a generalized homogeneous function of suitable combinations of only two variables (Binder 1981), $\tilde{P}$ being a 'scaling function',

$$P_L(\boldsymbol{m}) = L^{\beta/\nu}\tilde{P}(L/\xi, \boldsymbol{m}L^{\beta/\nu}) \qquad \xi \to \infty,\ L \to \infty, \quad L/\xi \quad \text{finite}. \tag{16}$$

From equation (16) we immediately conclude the standard finite size scaling relations (Privman 1991)

$$\langle|\boldsymbol{m}|\rangle = \int \mathrm{d}\boldsymbol{m}|\boldsymbol{m}|P_L(\boldsymbol{m}) = L^{-\beta/\nu}\tilde{f}_1(L/\xi) \tag{17}$$

$$\langle\boldsymbol{m}^2\rangle = \int \mathrm{d}\boldsymbol{m}\boldsymbol{m}^2 P_L(\boldsymbol{m}) = L^{-2\beta/\nu}\tilde{f}_2(L/\xi) \tag{18}$$

where $\tilde{f}_1$, $\tilde{f}_2$ are other scaling functions. From equation (13) we then obtain

$$T_c\chi' = L^{d-2\beta/\nu}[\tilde{f}_2(L/\xi) - (\tilde{f}_1(L/\xi))^2]. \tag{19}$$

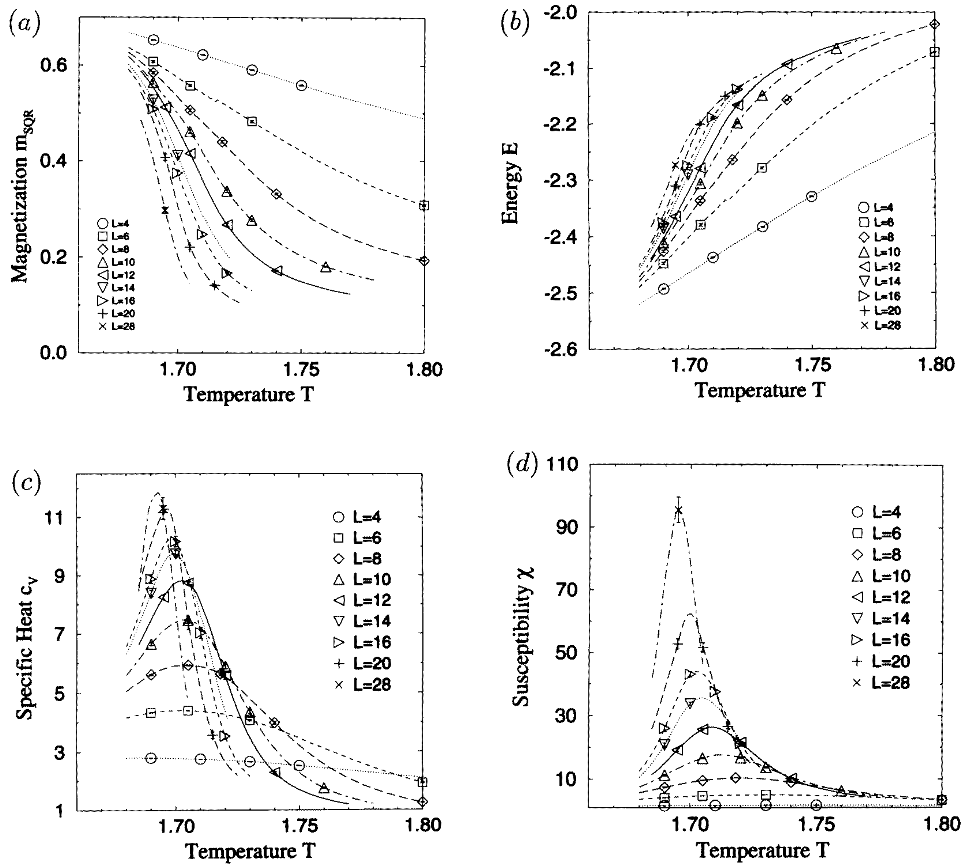


Figure 5. Energy E (a), root mean square magnetization m_{SQR} (b), specific heat C_v (c) and susceptibility χ' (d) plotted against temperature for $H_{RF} = 1$ and various lattice sizes L, with $4 \leqslant L \leqslant 28$, as indicated. The points indicate actual measured values, the curves have been constructed using histogram reweighting as described in the text.

Redefining a scaling function $\tilde{\chi}'(L/\xi)$ as follows

$$\tilde{\chi}'(L/\xi) = \tilde{f}_2(L/\xi) - [\tilde{f}_1(L/\xi)]^2 \tag{20}$$

we conclude from equations (19), (20) that $T_c\chi' = L^{d-2\beta/\nu}\tilde{\chi}'(L/\xi)$. Since finite size scaling of the susceptibility implies, on the other hand (Fisher 1971, Barber 1983)

$$T_c\chi' = L^{\gamma/\nu}\tilde{\chi}'(L/\xi) \tag{21}$$

it follows as a consequence of finite size scaling (Binder 1981) that

$$d - 2\beta/\nu = \gamma/\nu \tag{22}$$

which is nothing but the standard hyperscaling relation (Fisher 1974).

How can we apply then finite size scaling concepts to the RFIM and RFPM? Other cases where hyperscaling is violated are systems above their upper critical dimension d_u, and then a different form of finite size scaling holds where the correlation length ξ in equations (16)–(21) is replaced by the 'thermodynamic length' (Binder *et al* 1985, Binder 1985), and the exponent β/ν is replaced by $d/4$ (Zinn-Justin and Brézin 1985). Assuming here a second-order transition from the disordered phase to the ferromagnetic phase, hyperscaling is violated (equation (2)), but the resolution of the puzzle is again different.

First of all, we emphasize that irrespective of any finite size scaling assumption one can relate the finite size behaviour of the susceptibility and the disconnected susceptibilities at T_c to the power law decay of the corresponding critical correlations

$$g_{conn}(\boldsymbol{r}) \equiv [\langle \boldsymbol{S}_o \cdot \boldsymbol{S}_r \rangle_T - \langle \boldsymbol{S}_o \rangle_T \cdot \langle \boldsymbol{S}_r \rangle_T]_{av} \propto r^{-(d-2+\eta)} \quad T = T_c \tag{23}$$

$$g_{dis}(\boldsymbol{r}) \equiv [\langle \boldsymbol{S}_o \rangle_T \cdot \langle \boldsymbol{S}_r \rangle_T]_{av} \propto r^{-(d-4+\bar{\eta})} \quad T = T_c \tag{24}$$

recalling the standard definitions of the exponents η (Fisher 1974) and $\bar{\eta}$ (Schwartz 1985, Schwartz and Soffer 1985, 1986). The finite size behaviour now follows straightforwardly

$$T_c \chi \equiv \sum_r g_{conn}(\boldsymbol{r}) \propto \int_0^L r^{d-1} \mathrm{d}r\, g_{conn}(r) \propto L^{2-\eta} \tag{25}$$

and

$$T_c \chi_{dis} \equiv \sum_r g_{dis}(\boldsymbol{r}) \propto \int_0^L r^{d-1} \mathrm{d}r\, g_{dis}(r) \propto L^{4-\bar{\eta}}. \tag{26}$$

If one assumes scaling properties of these correlation functions

$$g_{conn}(\boldsymbol{r}) = r^{-(d-2+\eta)} \tilde{g}_{conn}(r/\xi) \tag{27}$$

$$g_{dis}(\boldsymbol{r}) = r^{-(d-4+\bar{\eta})} \tilde{g}_{dis}(r/\xi). \tag{28}$$

It follows that $T\chi \propto \xi^{2-\eta}$, $T\chi_{dis} \propto \xi^{4-\bar{\eta}}$, and then equations (25) and (26) can also be rewritten as

$$T_c \chi \propto L^{\gamma/\nu} \qquad T_c \chi_{dis} \propto L^{\bar{\gamma}/\nu} \tag{29}$$

Following the similar reasoning of Schwartz (1985) and Schwartz and Soffer (1985, 1986), we make use of the obvious inequalities

$$L^d [\langle \boldsymbol{m} \rangle_T]_{av}^2 \leqslant L^d [\langle \boldsymbol{m} \rangle_T^2]_{av} \leqslant L^d [\langle \boldsymbol{m}^2 \rangle_T]_{av}. \tag{30}$$

Since equation (14) says that the quantity in the middle of this inequality is the disconnected susceptibility, it follows that at T_c

$$L^{4-\bar{\eta}} \leqslant L^d [\langle \boldsymbol{m}^2 \rangle_{T_c}]_{av} = L^{d-2\beta/\nu} \tag{31}$$

where in the last step equation (18) was used (here $P_L(\boldsymbol{m}, T) = [P_L(\boldsymbol{m}, T, \{h_i\})]_{av}$ is the distribution function obtained after taking the quenched average over the random fields). Equation (31) is equivalent to the inequality $4 - \bar{\eta} \leqslant d - 2\beta/\nu$. We now assume that for 'reasonable' distributions $P_L(\bar{m}, T)$, with respect to the powers of L at T_c in equation (31) we should have an equality,

$$4 - \bar{\eta} = d - 2\beta/\nu \qquad \text{or} \qquad \bar{\gamma}/\nu + 2\beta/\nu = d \tag{32}$$

where in the last step equation (29) was invoked. Thus we conclude that *if* one assumes that 'L scales with ξ' (as written in equations (16)–(19)) it follows under fairly general assumptions that there is a kind of hyperscaling relation, but with $\bar{\gamma}$ replacing γ in the usual relation, equation (22).

Now it is possible to give a simple argument to derive a relation that shows that $\bar{\gamma}$ is much larger than γ, and confirms equation (3). Similar to reasonings presented by Schwartz (1985, 1991) and Schwartz *et al* (1991), we simply consider now one particular realization of the random field in a volume L^d. In one realization, there will typically be an excess of fields favouring one state of order $H_{RF} L^{-d/2}$, and hence a magnetization will result

$$\langle |\boldsymbol{m}| \rangle_T = H_{RF} \chi L^{-d/2} \tag{33}$$

where χ is the standard (connected) susceptibility. Assuming now at T_c that $\langle|\boldsymbol{m}|\rangle_{T_c} \propto L^{-\beta/\nu}$, $\chi_{T_c} \propto L^{\gamma/\nu}$ (equations (17) and (29)), we now get from equation (33)

$$-\beta/\nu = \gamma/\nu - d/2 \qquad d = 2(\beta+\gamma)/\nu. \tag{34}$$

Combining equations (32) and (34) readily yields

$$\bar{\gamma} = 2\gamma \tag{35}$$

or

$$\gamma/\nu + 2\beta/\nu = d - \gamma/\nu = d - (2-\eta) \tag{36}$$

which is nothing but equation (3). Using $\bar{\gamma} = \nu(4-\bar{\eta}) = 2\gamma = 2\nu(2-\eta)$ one also finds $\bar{\eta} = 2\eta$, of course (Schwartz 1991).

How can we then avoid the contradiction that was pointed out in equations (20)–(22)? The answer, simply, is that we have to abandon equations (19) and (20), because in the scaling limit $\tilde{f}_2(L/\xi) = [\tilde{f}_1(L/\xi)]^2$, and therefore $\tilde{P}(0, \boldsymbol{m}L^{\beta/\nu})$ is a delta function at positions $|[\langle \boldsymbol{m}\rangle_{T_c}]_{av}| = cL^{-\beta/\nu}$, where c is a constant. Since in the connected susceptibility the two leading terms then cancel, we have to include the leading correction to scaling when we consider the connected susceptibility. Thus

$$[\langle \boldsymbol{m}^2\rangle_{T_c}]_{av} = c^2 L^{-2\beta/\nu} + \hat{\chi} L^{\gamma/\nu - d}. \tag{37}$$

Our analysis thus implies that the order parameter distribution in finite systems at criticality has q sharp peaks at positions of magnitude $cL^{-\beta/\nu}$ and half width $L^{(\gamma/\nu-d)/2}$. In systems where hyperscaling holds, peak positions and width scale with the same power of L, while in our case the relative width of the peak vanishes in the scaling limit:

$$\text{relative width} \propto L^{(\gamma/\nu-d)/2+\beta/\nu} = L^{-\theta/2} = L^{-(1-\eta/2)} \to 0. \tag{38}$$

This behaviour also implies that unlike standard critical phenomena the fourth order cumulant (Binder 1981, 1992) which we define here as follows

$$g_L(T) = \frac{5}{2} - \frac{3}{2}\frac{[\langle(\boldsymbol{m}^2)^2\rangle_T]_{av}}{[\langle \boldsymbol{m}^2\rangle_T]_{av}^2} \tag{39}$$

does not have at T_c an intersection point with a nontrivial universal value, but rather the cumulant intersection should converge towards the trivial zero temperature fixed point value $g_L^\star(T \to 0) = 1$, as has been briefly announced by Eichhorn and Binder (1995).

4. Numerical results

Our first task is to locate the transition temperature T_c, and to characterize the order of the transition, putting into the analysis as few assumptions as possible. For Potts models, a popular characteristic to discuss the order of the transition is the cumulant V_E of the energy distribution (Challa *et al* 1986, Binder 1992)

$$V_E = 1 - [\langle \mathcal{H}^4\rangle_T]_{av}/3[\langle \mathcal{H}^2\rangle_T]_{av}^2. \tag{40}$$

This quantity is analysed in figure 6. It is seen that minima in this quantity do occur, but the depth of the minima (relative to the trivial limit $V_E(L \to \infty, T \neq T_c) = \frac{2}{3}$) rapidly decreases. Obviously, these data do not suggest a first-order transition with a latent heat, since the existence of a latent heat would imply that the depth of the minimum saturates for $L \to \infty$ at a finite value less than the trivial limit $2/3$. In view of figure 6(*b*), such a possibility is extremely unlikely.

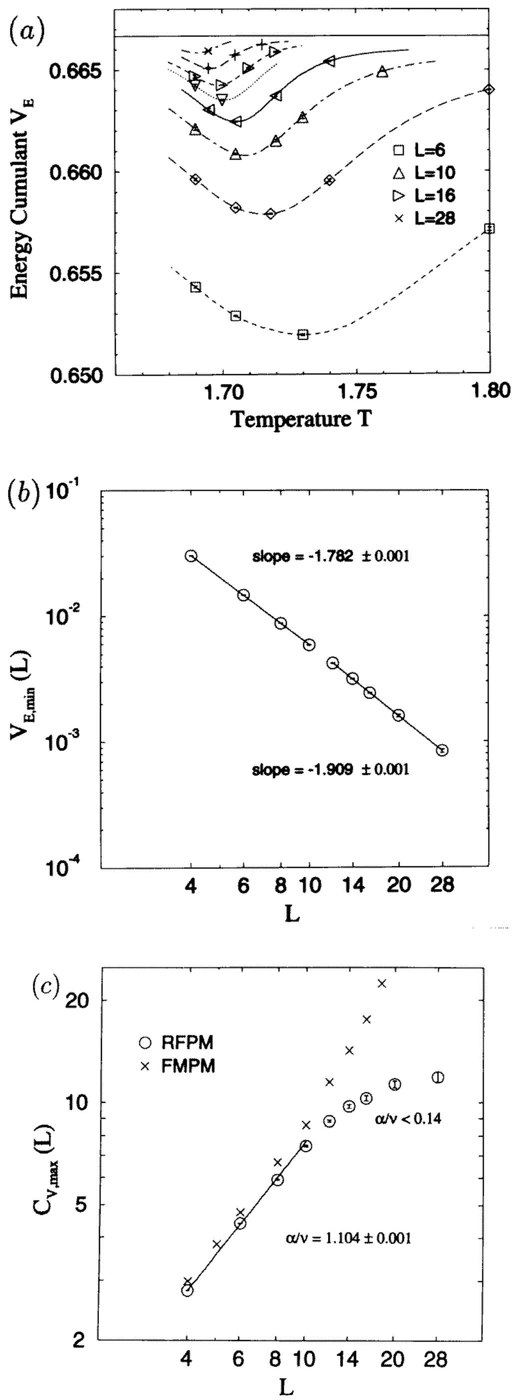


Figure 6. (*a*) Energy cumulant V_E plotted against temperature. Various sizes are indicated as shown in the figure. (*b*) Log–log plot of the difference $V_{E,min}(L) \equiv 2/3 - V_E(T_{min}, L)$ versus L. Straight lines indicate possible power law fits. (*c*) Log–log plot of the specific heat maxima $c_{V,max}(L)$ against L. Crosses indicate corresponding data of a pure ferromagnetic Potts model (FMPM) without random fields.

In principle, from the slope on the log–log plot figure 6(*b*) one could extract some estimate for the critical exponent α of the specific heat, assuming a second-order transition. But the direct analysis of the specific heat maxima shows that there is an obvious problem of crossover: for $L \lesssim 10$ the data are hardly distinct from those of the pure Potts model (which has a weak first-order transition, see e.g. Heermann 1979, Vollmayr *et al* 1993 for a discussion and references). While the pure Potts model bends over towards the exponent $d = 3$, which on the log–log plot $C_{v,max}(L)$ against L is a clear proof of the nonvanishing latent heat, in the random field case we see that the specific heat maxima bend over to a rather small slope, but it is even possible that they saturate at a finite maximum for $L \to \infty$ (we tentatively tried to estimate this maximum, obtaining $C_{v,max}(L \to \infty) \approx 12.8$, but the error of this estimate is rather uncertain). Since the critical behaviour of the specific heat is notoriously difficult to estimate from Monte Carlo work (Binder and Heermann 1992), we do not attempt to extract any critical exponent estimates from figure 6.

Both for second-order transitions (Binder 1981,1992) and for first-order transitions (Vollmayr *et al* 1993) in pure systems a study of the fourth-order cumulant (equation (39)) and its intersections for different linear dimensions L is a good method to estimate T_c. Figure 7 demonstrates, however, that for the RFPM this intersection property does not hold.

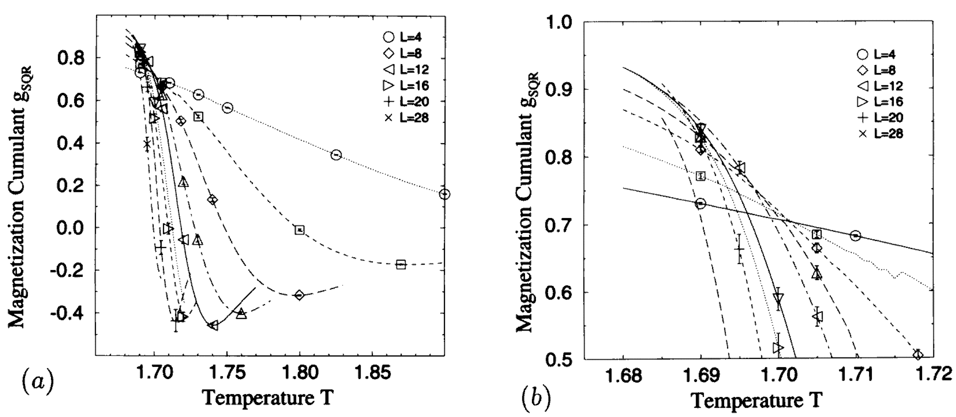


Figure 7. Cumulant $g_L(T)$ (equation (39)) plotted against temperature for various sizes L. Part (*a*) shows an overall view, part (*b*) presents a detailed view of the region near T_c.

The larger the lattice sizes used, the more the intersection points move upwards towards the trivial values, $g_{L\to\infty}(T_c) = 1$. Of course, in the light of the discussion presented in the previous section this behaviour is expected. Again, the conclusion is that the data are in a crossover regime, and the asymptotic region is presumably only reached for much larger choices of L than were available. Note also the minima $g_{L,min}(T = T_{min})$, which are reminiscent of the behaviour of the pure Potts model (Vollmayr *et al* 1993). In the latter case, these minima diverge to $g_{L,min}(T = T_{min}) \to -\infty$ as $L \to -\infty$, since the transition is of first-order in the pure case, while in the present case the depth of the minima does not increase for large L.

Since for the present problem the cumulant intersection technique fails, we have followed Ferrenberg and Landau (1991) in applying a different recipe to locate T_c, namely we extrapolate various characteristic temperatures $T_c(L)$ as a function of $1/L^{1/\nu}$, trying a linear extrapolation with as many quantities simultaneously, in order to obtain reliable estimates for both T_c and $1/\nu$. Such characteristic temperatures that we have used are the position of

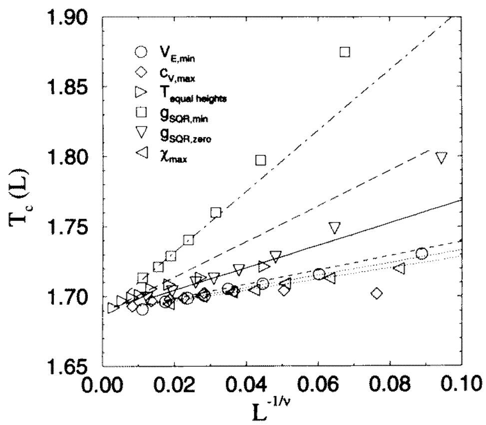


Figure 8. Extrapolation of $T_c(L)$ against $L^{-1/\nu}$, using various characteristic temperatures $T_c(L)$: minima of order parameter cumulant (squares) and energy cumulant (circles), maxima of specific heat (diamonds) and susceptibility (triangles pointing to the left). Also zeros of the order parameter cumulant (triangles pointing down) and temperatures where the energy distribution has two peaks of equal height (figure 9(a)) are used. Result of simultaneous straight lines fit is $T_c = 1.6900 \pm 0.002$ and $1/\nu = 1.535 \pm 0.009$.

the minima of the energy cumulant ($V_{E,min}$) and the order parameter cumulant ($g_{SQR,min}$), as well as the position where $g_L(T) = 0$, as well as maxima of specific heat ($C_{v,max}$) and susceptibility $\chi'(\chi_{max})$, see figure 8. Also the temperature $T_{equal\ heights}$ was used where both peaks of the energy distribution have equal heights (figure 9(a)). Note that this distribution does not develop a deep minimum between the peaks, and although we do have a mild double-peak structure in $P(E)$ we do not have a peak for the disordered phase in the order parameter distribution (figure 9(b)) at criticality. This method seems to yield a rather good estimate of T_c, namely $T_c \approx 1.690$; however, the error bars resulting for T_c and $1/\nu$ from this fit (as quoted in the figure caption of figure 8) cannot be taken seriously, since this analysis does not account for the systematic problem that at least part of the data (if not all of them!) are not yet in the regime of large enough L where finite size scaling without corrections holds. In fact, if we fit the individual data in figure 8 separately, we still get a good estimate for T_c, $T_c \approx 1.6909 \pm 0.0006$, but wildly fluctuating estimates for $1/\nu$ (Eichhorn 1995).

Using this estimate of T_c, we could now check the finite size scaling of the various moments $[\langle|\boldsymbol{m}|^k\rangle_{T_c}]_{av}$. In order to eliminate errors from inaccuracy of T_c we rather use the data at the temperatures of the susceptibility maxima, $T_c(L)$. Assuming a leading behaviour $[\langle|\boldsymbol{m}|^k\rangle_{T_c(L)}]_{av} \propto L^{-k\beta/\nu}$ and disregarding the effect of corrections such as postulated in equation (37), the data for $L \geqslant 12$ are compatible with an exponent $\beta/\nu = 0.34 \pm 0.01$. In this context it is interesting to remember that Rieger and Young (1995) found $\beta/\nu \approx 0$ for the RFIM, raising the possibility of a first-order transition without latent heat. Our data for the RFPM do not support such an interpretation. However, again note the caveat that it is possible that the asymptotic regime has not yet been reached in the range $12 \leqslant L \leqslant 28$. Furthermore, it must be noted that a log–log plot of $[\langle|\boldsymbol{m}|^k\rangle_{T_c}]_{av}$ is compatible with a significantly smaller estimate, namely $\beta/\nu \approx 0.18 \pm 0.01$. Similar ambiguities concern our estimates for the exponents characterizing the size dependence of connected and disconnected susceptibilities (figure 10(b)).

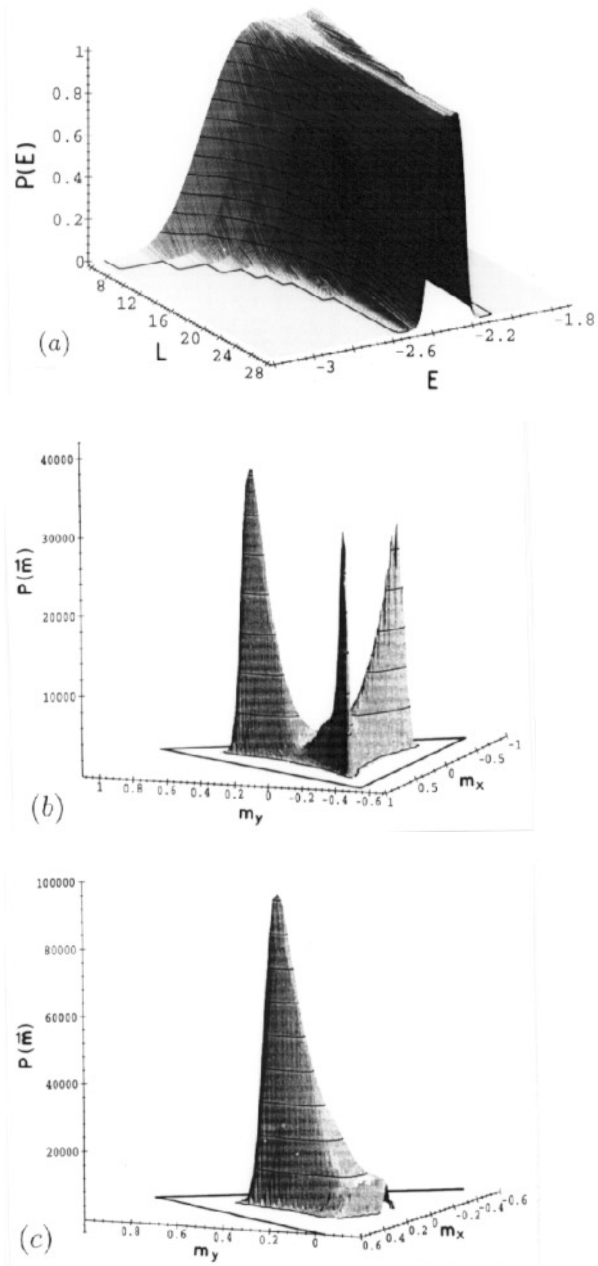


Figure 9. (*a*) Energy distribution $P(E)$ shown as function of energy E per spin end linear dimension L, at temperatures $T_{equal\ heights}$ where both peaks are equal, and normalized such that these peaks have height unity. (*b*) Distribution of the order parameter $P(\boldsymbol{m})$ for $L = 28$ at $T = 1.6948$ (the position of the susceptibility maximum). (*c*) Same as (*b*) but all three states projected on a single state (1/0), in order to symmetrize the distribution and smooth out fluctuations.

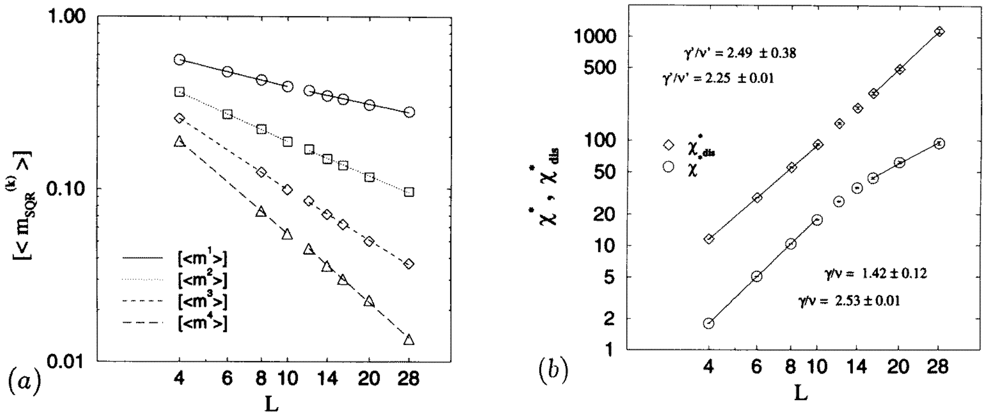


Figure 10. (*a*) Log–log plot of the first four moments of the root mean square magnetization $m_{SQR} = |\boldsymbol{m}|$ versus linear dimension at $T_c(L)$ defined by the susceptibility maximum. Note that slight curvature occurs on this plot, as exemplified by fitting straight lines with different slopes for the data with $L \leqslant 10$ and $L \leqslant 12$, respectively. (*b*) Log–log plot of the maximum value $\chi^\star$ of $\chi(T_c(L))$, circles, and of the disconnected susceptibility $\chi^\star_{dis}$ at the same temperature $\{\chi^\star_{dis} = \chi_{dis}(T_c(L))\}$ versus the linear dimension L. Note that for $L \leqslant 10$ effective exponents $\gamma/\nu \approx 2.53$ (connected) and $\bar{\gamma}/\nu \approx 2.25$ could be extracted, which presumably are not meaningful. For $L \geqslant 12$ the straight lines shown have the slopes $\gamma/\nu = 1.42$ and $\bar{\gamma}/\nu = 2.49$, respectively, but the real accuracy of these estimates is still rather uncertain due to crossover problems.

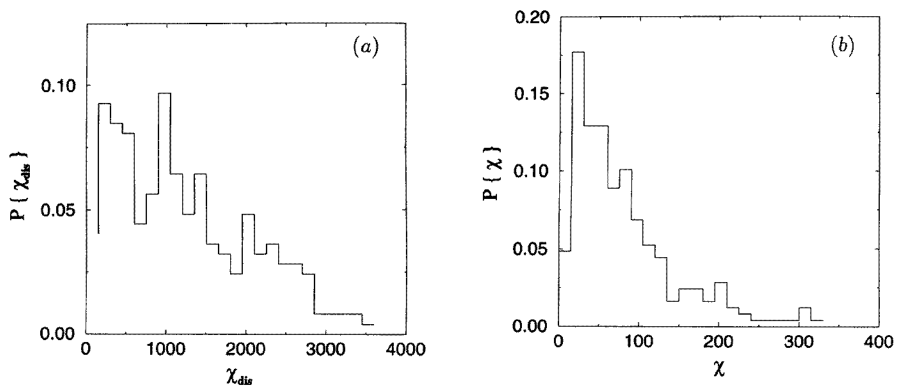


Figure 11. Distribution $P(\chi_{dis})$ of the disconnected susceptibility (*a*) and distribution $P(\chi)$ of the connected susceptibility (*b*) for $L = 28$ at the temperature $T_c(L) = 1.6948$ of the susceptibility maximum. These distributions were generated from a sample of 248 random fields. Note the different abscissa scales in parts (*a*) and (*b*).

Taking these estimates in the above scaling relations, equations (2), (3) and (32) we obtain $\bar{\gamma}/\nu + 2\beta/\nu = 2.49 + 0.68 = 3.17$ or $2.49 + 0.36 = 2.85$, which should be $d = 3$. Also the relation $\bar{\gamma}/\nu = 2\gamma/\nu = 2.84$ is at best very roughly fulfilled. We suspect that the real accuracy of our exponent estimates is at best 10–20 %, and more accurate data and larger system sizes clearly would be very desirable. The necessary computing resources for such a task are still orders of magnitude larger than what was available to us, however. Nevertheless we feel that our numerical data provide at least qualitative evidence for the scaling description presented in the previous section. Further evidence for the unusual character of this transition comes from a study of the distribution function of the susceptibilities (figure 11). While $P(\chi_{dis})$ is very broad and does not indicate any tendency of χ_{dis} for 'self-averaging' despite the sharpness of $P(m)$, figure 9(*b*), $P(\chi)$ has its peak at rather small values but there is a clearly developed tail to much larger values. Similar findings were also reported for the RFIM (Rieger and Young 1993). But again a much larger sample of random field configurations would be necessary to clarify the analytic nature of these distributions.

5. Conclusions

In this paper we have presented data for the three-dimensional Potts ferromagnet with $q = 3$ states exposed to random fields of intermediate strength (i.e., $H_{RF} = J$ was chosen). Consistent with our phase diagram scenario, figure 2(*b*), we find that for small linear dimensions the random field has only small effects, and the behaviour is very similar to the pure Potts ferromagnet without random fields (figure 2(*b*) implies that for weak enough random fields the transition stays first-order, and only for fields exceeding a threshold value—corresponding to the lower line of tricritical points in figure 2(*b*)—does one have second-order transitions). All our data are compatible with a standard second-order transition from the disordered phase to the ferromagnetic phase. This finding thus disagrees with the result of Goldschmidt and Xu (1985, 1986), that the transition should be first-order throughout, and also does not provide any evidence for the idea (e.g. de Almeida and Bruinsma 1987, Mézard and Monasson 1994, de Dominicis *et al* 1995) that there should intervene a glass-like phase between the disordered and ferromagnetic phases. However, the results are compatible with the ideas of Schwartz (1985, 1991), Schwartz and Soffer (1985,1986), Schwartz *et al* (1981) and Gofman *et al* (1993) that there should be critical behaviour with a two-exponent scaling, corresponding to the 'modified dimensional reduction' (equations (2) and (3)). Although we cannot extract accurate exponent estimates yet, the trend of our numerical observations is plausible and goes in the right direction expected from this scenario. An important part of our paper is that we have worked out the consequences of this scenario for finite size scaling in detail (section 3). Our numerical data support this description at least qualitatively. There is a clear need to study both much larger system sizes and larger samples of the random field configuration, but this requires either significantly more efficient simulation algorithms or significantly faster computers. Recalling that the model may have applications to experiments on orientational glasses, a search for suitable experimental realizations of this 'universality class' may be also rewarding.

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